Reply to the comment on the transcendental method

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COMMENT

Reply to the comment on the transcendental method

S M Perovich†

Department of Nuclear Engineering, University of Montenegro, Montenegro, Yugoslavia

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Abstract. Using an unconventional mathematical approach and analysis of the precision of solution by a new formula for the representation of number in the transcendental method, we indicate that the objection given in the comment most probably shows a lack of understanding and misapprehension. Undoubtedly, the transcendental equation of the neutron slowing down has an analytical closed-form solution. This solution is exact according to the new closed-form representation of numbers with the desired accuracy. The numerical results, obtained for different magnitudes of the transcendental parameter $B(u)$, support the validity and basic principles of the transcendental method.

1. On the nature of the analytical closed-form solution to the transcendental equation of a neutron slowing down

The comment (Miranović 1995) confines its attention to the transcendental equation described in equation (61)P (Perovich 1992). This equation, after some elementary simplification, takes the form

$$\begin{aligned} &Y(u) = B(u)\exp(Y(u)) \\ &B(u) > 0 \qquad B(u) < 1/\exp(1) \qquad Y(u) < 1 \end{aligned} \tag{1}$$

where

$$Y(u) = Z_1(u)B_0(u) \qquad B(u) = B_0(u)B_1(u)$$

$B_0(u)$, $B_1(u)$ and $Z_1(u)$ are given in equations (62)P and (63)P, respectively (Perovich 1992).

Fortunately, the transcendental approach yields a general methodology for computing $Y(u)$, thus the solution of equation (1) can be written in the closed-form representation

$$Y(u) = \ln(F_-(y,u)/F_-(y+1,u)) \qquad \text{for large } y \tag{2}$$

or, consequently, for fixed u, the number Y takes the following form

$$Y = \ln\left\{\frac{\sum_{n=0}^{[y]}(-B)^n(y-n)^n/n!}{\sum_{n=0}^{[y+1]}(-B)^n(y+1-n)^n/n!}\right\} \qquad \text{for large } y \tag{3}$$

† Present address: PO Box 120, 85310 Budva, Montenegro, Yugoslavia.

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since the dissonance function $F_{-}(y, u)$ is defined as

$$F_{-}(y,u) = \begin{cases} \sum_{n=0}^{[y]} (-B(u))^n (y-n)^n/n! & \text{for } y > 0 \\ 0 & \text{for } y < 0 \end{cases} \tag{4}$$

(equation (68)P, where $\Phi(B_0(u)y, u) = F_{-}(y, u)$ and $\Phi_0 = 1$) or, more explicitly, as

$$F_{-}(y,u) = \sum_{n=0}^{\infty} (-B(u))^n (y-n)^n H(y-n)/n! \tag{4$'$}$$

where H is the Heaviside's unit function defined as

$$H(y-n) = \begin{cases} 1 & \text{for } y > n \\ 0 & \text{for } y < n. \end{cases}$$

By differentiation from equations (4) and (4′) we find

$$\partial F_{-}(y,u)/\partial y = \sum_{n=1}^{[y]} (-B(u))^n \frac{(y-n)^{n-1}}{(n-1)!} + \sum_{n=0}^{[y]} (-B(u))^n \frac{(y-n)^n}{n!} \frac{\partial H(y-n)}{\partial y}.$$

Now, after a simple modification, the above equation can be written in the form:

$$\partial F_{-}(y,u)/\partial y = -(B(u))F_{-}(y-1,u) + \sum_{n=0}^{[y]} (-B(u))^n (y-n)^n \delta(y-n)/n! \tag{5}$$

(since $\partial H(y-n)/\partial y = \delta(y-n)$). Furthermore, in view of $(y-n)\delta(y-n) = 0$ the statement $(y-n)^m \delta(y-n) = (y-n)^{n-1}[(y-n)\delta(y-n)] = 0$, for $n \geqslant 1$, follows trivially. (See, for instance, Louis Maisel (1971), section 2.3: Dirac delta function.) Thus, we find that, for $y > 0$, equation (5) takes the following form

$$F_{-}(y-1,u) = (-1/B(u))\partial F_{-}(y,u)/\partial y.$$

In fact, this result can also be obtained by differentiating (4) for $[y] = K$, $(K < y < K+1)$. Thus, equation (5) is reconciled with the statement following equation (19)P. From a theoretical point of view, the solution (3) for Y can be found with an arbitrary order of accuracy, taking an appropriate value of y. The relation between y and precision of solution P is given graphically. The nature of precision $P(y, u)$ as a function of lethargy y can be seen very well in figure 1, which is plotted using the right-hand side of the following equation

$$P(y,u) = -\lg|G(y,u)| \tag{6}$$

where

$$G(y,u) = Y(u) - B(u)\exp(Y(u))$$

for different values of y. Numerical results obtained for a solution precision $P(y, u)$ are presented in figure 1 for several values of $B(u)$ (the figure is obtained using the

MATHEMATICA program). Let us note that a similar tendency has also been observed for other values of B. According to the author's knowledge, formula (3) for the presentation of the number Y is a new one, and shows that number of accurate digits in the structure of Y depends on the lethargy y. In other words, the choice of y controls the number of accurate digits for the constants Y, and satisfies the error criterion. As is well known the number of accurate digits in the numerical structure of constant Y is practically determined by the physical requirements of the neutron's slowing down process, independently of the theoretical idea of infinite precision. Note that the applications of the exact numerical structure (3) have been limited by the unreachable limit of an infinite precision for constant Y (similarly as for constants $\exp(1)$, π, $\sqrt{2}$, etc). For these reasons, the practical applicable formula for Y takes the form

$$Y(u) = \ln\left(\frac{F_{-}(y,u)}{F_{-}(y+1,u)}\right) \qquad \text{for } y \geqslant y_g \tag{7}$$

where y_g is the value of the lethargy y when the error function defined as

$$G(y,u) = Y(u) - B(u)\exp(Y(u)) \tag{8}$$

satisfies the inequality

$$|G(y,u)| \leqslant g \qquad \text{for } y \geqslant y_g \tag{9}$$

where g is an arbitrary small positive number and, for large y, g vanishes. Large y, in the theoretical sense, denotes the infinite large value of y which still contains the physical individuality of the lethargy, and satisfies the conventional requirements of mathematical infinity.

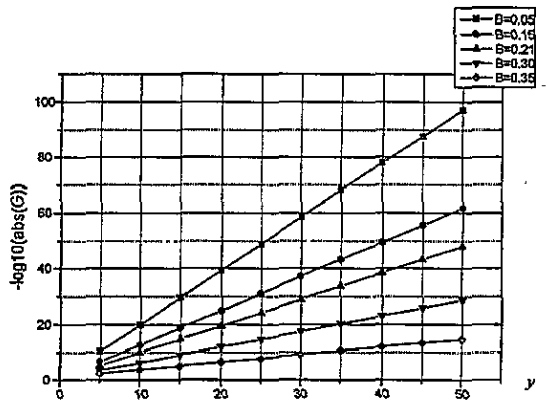

Figure 1. Relationship between the solution precision and lethargy for different parameters B.

2. Concerning the genesis of analytical closed-form solution (2)

The transcendental equation (1) can be identified with an integral equation of type

$$F_{-}(y,u) = B(u)\int_{y-1}^{\infty} F_{-}(t,u)\,\mathrm{d}t \qquad \text{for } y > 1$$

(equation (67)P, where $F_-(y,u) = \Phi(B_0(u)y, u)$) or, in the form

$$F_-(y,u) = F_-(y_0,u) - B(u)\int_{y_0}^{y} F_-(t-1,u)\,\mathrm{d}t \qquad \text{for } y > y_0 \tag{10}$$

where

$$F_-(y_0,u) = B(u)\int_{y_0}^{\infty} F_-(t-1,u)\,\mathrm{d}t. \tag{11}$$

Substituting expression (5) for $F_-(t-1,u)$ on the right-hand side of equation (10), we obtain

$$F_-(y,u) = F_-(y_0,u) + F_-(y,u) - F_-(y_0,u)$$

since

$$\int_{y_0}^{y} \sum_{n=0}^{[t]} (-B(u))^n (t-n)^n \delta(t-n)\,\mathrm{d}t/n! = 0$$

according to the following statement (Maisel 1971): *If a function $D(t)$ is continuous at $t = t_0$ then, for $a < b$, we find*

$$\int_a^b D(t)\delta(t-t_0)\,\mathrm{d}t = \begin{cases} D(t_0) & \text{for } a < t_0 < b \\ 0 & \text{for } t_0 < a \text{ or } t_0 > b. \end{cases}$$

Now, we have completed our proof that dissonance function $F_-(y,u)$ satisfies integral equation (10). Furthermore, the solution $F_-(y,u)$ is unique. On the other hand, the particular solution of the form

$$F_{-p}(y,u) = \frac{\exp(-Y(u)y)}{1-Y(u)} \tag{12}$$

satisfies integral equation (10), under one single condition that $y \in D_y$, *where* D_y is the domain of large (or near infinite) lethargies y. Namely, from equation (12), we obtain

$$\partial F_{-p}(y,u)/\partial y = -B(u)F_{-p}(y-1,u). \tag{13}$$

After substituting equations (12) and (13) in equation (10), we find

$$F_{-p}(y,u) = F_-(y_0,u) + F_{-p}(y,u) - F_{-p}(y_0,u).$$

Accordingly, the above equality is satisfied provided that

$$F_-(y_0,u) = F_{-p}(y_0,u). \tag{14}$$

Equation (14) is exact (infinite precision) only for large (or near infinite) y_0. For other values of y_0 (finite values), the equalization (14) is exact with a limited number of accurate digits (finite precision). Thus, substituting the results (4) and (12) for $F_-(y,u)$ and $F_{-p}(y,u)$, respectively, in equation (14) we obtain

$$\sum_{n=0}^{[y]} (-B(u))^n (y-n)^n/n! = \frac{\exp(-Y(u)y)}{1-Y(u)} \qquad \text{for large } y. \tag{15}$$

Of course, for an arbitrary finite value of lethargy y_f in the above equation there should be an approximation sign, and then we have

$$\sum_{n=0}^{[y_f]}(-B(u))^n(y_f-n)^n/n! \approx \frac{\exp(-Y(u)y_f)}{1-Y(u)}.$$

Since $B(u) = Y(u)\exp(-Y(u))$, then from equation (15), we obtain

$$(1-Y(u))\sum_{n=0}^{[y]}(-1)^n Y^n(u)\exp(Y(u)(y-n))(y-n)^n/n! = 1 \qquad \text{for large } y$$

or, more explicitly, in the form

$$(1-Y(u))\sum_{n=0}^{[y]}((-Y(u))^n(y-n)^n/n!\sum_{m=0}^{\infty}(Y(u))^m(y-n)^m/m! = 1 \qquad \text{for large } y. \quad (16)$$

After series expansion on the left-hand side, the above equation, for fixed values of the lethargy u, can be written as

$$(1-Y)\left[\sum_{m=0}^{\infty}Y^m y^m/m! - \sum_{m=0}^{\infty}Y^{m+1}(y-1)^{m+1}/(m!1!) + \sum_{m=0}^{\infty}Y^{m+2}(y-2)^{m+2}/(m!2!) + \cdots \right.$$
$$\left. + (-1)^K\sum_{m=0}^{\infty}Y^{m+K}(y-K)^{m+K}/(m!K!) + \cdots + \cdots\right] \qquad \text{for large } y.$$

From the above expansion, clearly, term of Y^M takes the following form:

$$(y^M/M! - y^{M-1}/(M-1)!) - ((y-1)^M/(M-1)! - (y-1)^{M-1}/(M-2!))$$
$$+ ((y-2)^M/((M-2)!2!) - (y-2)^{M-1}/((M-3)!2!)) + \cdots$$
$$+ (-1)^M((y-M)^M/(M!) - (y-M)^{M-1}/(M-1)!)$$

and, finally, on the left-hand side of equation (16) we obtain

$$\sum_{M=0}^{[y]} Y^M\left[\sum_{n=0}^{M}\frac{(-1)^n(y-n)^M}{(M-n)!n!} - \sum_{n=0}^{M-1}\frac{(-1)^n(y-n)^{M-1}}{(M-1-n)!n!}\right] \qquad \text{for large } y.$$

Thus, equation (16) takes the form

$$\sum_{M=1}^{[y]} Y^M(u)\sum_{n=0}^{M}\frac{(-1)^n(y-n)^{M-1}}{(M-n)!n!}(y-M) = 0 \qquad \text{for large } y \qquad (17)$$

(since for $M = 0$ we have: $1(y/y) = 1$). Note that the equality in (17) holds if and only if

$$\sum_{n=0}^{M}\frac{(-1)^n(y-n)^{M-1}}{(M-n)!n!} = 0 \qquad \text{for } \forall M \geqslant 1,\ M = 1, 2, 3, \ldots. \qquad (18)$$

By using mathematical induction we can simply show that equation (18) is satisfied for any positive integer $M \geqslant 1$. Using this approach of mathematical induction we have a complete proof for the existence of equality (15).

Finally, it is clear that particular solution, for large y, satisfies equation (10) according to the infinite precision of equalization (14). Then, applying a unique solution principle, we have

$$F_{-}(y,u) = F_{-p}(y,u) \qquad \text{for large } y (y > y_0) \tag{19}$$

or

$$\frac{F_{-}(y,u)}{F_{-}(y+1,u)} = \frac{F_{-p}(y,u)}{F_{-p}(y+1,u)} = \exp(Y(u)) \qquad \text{for large } y \tag{20}$$

and

$$Y(u) = \ln\left\{\frac{F_{-}(y,u)}{F_{-}(y+1,u)}\right\} \qquad \text{for large } y. \tag{21}$$

So, equation (1) can be solved analytically. Furthermore, it is perfectly clear that if equality (15) is true, then equations (2), (3) and (21) become exact, and the number Y can be obtained with infinite precision (exact solution). Obviously, formula (21) represents an analytical *closed-form solution of transcendental equation (1). Solution (21) is exact, because for* $Y(u)$ we have infinite precision. Clearly, the number of accurate digits in the numerical structure of Y is determined by the degree of accuracy in the equalization (19). Namely, equation (19) becomes exact for large y, and then we have infinite precision for number Y. For any finite values of y we have limited accuracy (finite precision) defined by equation (9). Let us note that numerical calculation of $Y(u)$ is practically possible by using the formula (7) in the following form:

$$Y = \ln\left\{\frac{\sum_{n=0}^{[y]}(-B)^n(y-n)^n/n!}{\sum_{n=0}^{[y+1]}(-B)^n(y+1-n)^n/n!}\right\} \qquad \text{for } y \geqslant y_g \tag{22}$$

where y_g is defined in the section 1.

At the end of this section I must emphasize that the dissonance functions $F_{-}(y,u)$ are not polynomials because of the Heaviside unit function and the Dirac delta function in the structure of derivatives. For example, from equation (5) we have

$$\partial^2 F_{-}(y,u)/\partial^2 y = \sum_{n=2}^{\infty}(-B(u))^n\frac{(y-n)^{n-2}}{(n-2)!}H(y-n) + 2\sum_{n=1}^{[y]}(-B(u))^n(y-n)^{n-1}\frac{\delta(y-n)}{(n-1)!}$$
$$+\sum_{n=0}^{[y]}(-B(u))^n\frac{(y-n)^n}{n!}(\partial\delta(y-n)/\partial y). \tag{23}$$

They are polynomial dissonance functions (as one class of the special functions). Consequently the analysis in the comment requires a very subtle justification.

3. Conclusions

First and foremost, there are serious doubts about the commentator's understanding of the relationship between an exact analytical closed-form solution (equation (3) and equation (73)P for large y), and a suitable approximate analytical closed-form solution, for number Y (equation (7) and equation (73)P for finite values of y). Obviously, the commentator misunderstood that equations (7) and (73)P, for any finite value of the lethargy y, represent the exact solutions in the transcendental method theory. However, equations (7) and (73)P, for finite values of y are approximate analytical closed-form solutions which give the numerical results with limited accuracy. On the other hand, in the transcendental method theory, formula (3) represents the numerical structure of the constants Y with infinite precision (unlimited accuracy), while formula (22) determines the constants Y with limited accuracy. In an exact way, the number of accurate digits, in the practical applicable Y, is in accordance with the physical requirements of exactness. Accordingly, the final form of the solution for the collision density, described in equation (65)P, continues to stay in a domain of an analytical closed form, regardless of the number of accurate digits in the numerical structure of constants Y (equations (6) and (22)), which are obtained by computer calculation. Namely, between the symbol Y represented in equation (3), and a suitable closed-form representation for number Y in equation (22), there exists a qualitative distinction regarding theoretical exactness. It is analogous to the essential distinction between the constant π and a finite number of known accurate digits in the numerical structure of the number π (5000 digits of accuracy obtained in the program MATHEMATICA, for instance).

The key to the problem is in the fact that the degree of accuracy in equalization (15) simulates the nature of solution precision $P(y, u)$. Thus, when equation (15) becomes exact (for large value of y) then we have infinite precision for number Y and, consequently, we have the exact solution of equation (1). On the other hand, the commentator's argument deals with the relationship between functions $F_{-}(y, u)$ and $F_{-p}(y, u)$ but only for finite y, when we have limited accuracy for equalization (15). In this way, from the obvious inequality $F_{-}(y) \neq F_{-p}(y)$ (or $Y(u)_{\text{in Eq.(3)}} \neq Y(u)_{\text{in Eq.(22)}}$) for finite values of y, the commentator concludes incorrectly that Y and collision density $\Phi(y, u)$, represented in equation (3) and in equation (65)P, respectively, are not exact solutions! (Analogously, according to the commentator's logic, from the obvious inequality $\pi \neq 3.14$ (as $Y(u)_{\text{in Eq.(3)}} \neq Y(u)_{\text{in Eq.(22)}}$), directly follows the statement that number π (as number Y) is not exact—and, consequently, that all expressions which contain the number π (circumference, volume of sphere etc), are not exact.) I believe that the commentator still does not doubt the exactness of the number π, but then, consequently, if the commentator accepts the number π as exact, why not accept the number Y as exact, as well. It is obviously that the commentator's approach is a tautologism because of the fact that the inequality $F_{-}(y) \neq F_{-p}(y)$ for a finite value of lethargy y, in the transcendental method theory, follows directly. Let us note that the commentator fails to see that the degree of accuracy in equalization (15) uniquely determines the number of accurate digits for Y, all the way to the infinite precision. This implies that the commentator does not have a clear idea about what finite y is, and what large y is in the transcendental method theory. Thus, the commentator, in his attempt to attain infinite precision for number Y (exact solution), forgets the domain of large y! Since, according to the transcendental method theory, it cannot give a valid result, he blames the transcendental method! Finally, if we agree that number Y in equation (3) has infinite precision, then we must accept the fact that solutions (3) and (73)P, for large y, are exact, in the theoretical sense. (For instance, the numbers Y are as exact as the numbers exp(1),

π, $\sqrt{2}$, etc.) Accordingly, the commentator's objection has no relevant meaning for the basic principles of the transcendental method and, most likely, the commentator's problem is the philosophical problem of the unreachable limit of infinite precision for the numbers like Y, $\exp(1)$, π, $\sqrt{2}$ etc.

Furthermore, the analytical closed-form representation (22) (and equation (6)) gives impressive results (see figure 1 and table 1), which, consequently, suggest that my method works. In this way, any doubts about the validity of applications of the transcendental method become scholastic speculations.

In accordance with the nature of dissonance function $F_-(y,u)$, the analysis of their derivatives in the comment requires a very subtle justification for any integer lethargy values $y(y = M, M = 1, 2, 3, \ldots)$. This can be seen by writing $\partial^k F_-(y,u)/\partial y^k$ for $M \leqslant y \leqslant M+1$ in the form

$$\partial^k F_-(y,u)/\partial y^k = \sum_{m=0}^{M} (-1)^m B^m(u) \delta^{(k-1-m)}(y-m) \qquad \text{for } k \geqslant M+1. \tag{24}$$

Formula (24) is simply derived. For instance, the function $F_-(y,u)$ (equation (4′)) for lethargy $2 \leqslant y < 3$ takes the form

$$F_{-2}(y,u) = H(y) - B(u)(y-1)H(y-1) + \frac{B^2(u)(y-2)^2}{2!} H(y-2). \tag{25}$$

After differentiation, we find

$$\partial F_{-2}(y,u)/\partial y = \delta(y) - B(u)H(y-1) + B^2(u)(y-2)H(y-2)$$

(since $(y-n)^n \delta(y-n) = 0$, for $n \geqslant 1$ and $\mathrm{d}^{(k)} H(y-n)/\mathrm{d}y^k = \delta^{(k)}(y-n)$ (Maisel 1971), and

$$\partial^3 F_-(y,u)/\partial y^3 = \delta''(y) - B(u)\delta'(y-1) + B^2(u)\delta(y-2).$$

Or, generally

$$\partial^k F_{-2}(y,u)/\partial y^k = \sum_{m=0}^{2} (-1)^m B^m(u) \delta^{(k-1-m)}(y-m) \qquad \text{for } k \geqslant 3. \tag{26}$$

Similarly, for $3 \leqslant y < 4$ the function $F_-(y,u)$ takes the following form:

$$F_{-3}(y,u) = F_{-2}(y,u) - B^3(u)\frac{(y-3)^3}{3!} H(y-3). \tag{27}$$

After differentiation, the above equation becomes

$$\partial^k F_{-3}(y,u)/\partial y^k = \sum_{m=0}^{2} (-1)^m B^m(u) \delta^{(k-1-m)}(y-m) - B^3(u)\delta^{(k-4)}(y-3) \qquad \text{for } k \geqslant 4$$

or

$$\partial^k F_{-3}(y,u)/\partial y^k = \sum_{m-0}^{3} (-B(u))^m \delta^{(k-1-m)}(y-m) \qquad \text{for } k \geqslant 4.$$

Table 1. For $B(u) = 0.17$. These results are obtained using the MATHEMATICA program. Let us note that a similar tendency has also been observed for other values of $B(u) < 1/\exp(1)$.

M	Y	P_-
10	0.209 652 377 6	11.7544
20	0.209 652 377 677 375 752 685 4	23.0148
30	0.209 652 377 677 375 752 685 411 045 492 259	34.2751
40	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133	45.5355
50	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992	56.7959
60	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992 366 422 464 74	68.0563
70	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992 366 422 464 748 106 611 532 5	79.3167
80	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992 366 422 464 748 106 611 532 546 262 815 29	90.557
90	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992 366 422 464 748 106 611 532 546 262 681 529 155 026 544 64	101.837
100	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992 366 422 464 748 106 611 532 546 262 681 529 155 026 544 641 510 377 542 81	113.098
200	0.209 652 377 677 375 752 685 411 045 492 259 182 117 234 133 891 279 50 992 366 422 464 748 106 611 532 546 262 681 529 155 026 544 641 510 377 542 812 386 658 664 001 589 206 481 912 547 426 853 691 967 439 436 661 775 793 364 724 124 019 218 399 086 669 939 456 856 985 801 346 004 055 838 516 501 158 169 156 529 795 506 3	225.702

Repeating the application of this method for $M \leqslant y < M + 1$, we find

$$\partial^k F_{-M}(y,u)/\partial y^k = \sum_{m=0}^{M-1} (-B(u))^m \delta^{(k-1-m)}(y-m) + (-1)^M B^M(u) \delta^{(k-1-M)}(y-M)$$
$$\text{for } k \geqslant M+1.$$

Finally, for $M \leqslant y < M + 1$, formula (24) is obtainable.

Equation (24) for $y = M$ (integer lethargy's values) takes the form

$$\partial^k F_-(y,u)/\partial y^k|_{y=M} = \sum_{m=0}^{M} (-1)^m B^m(u) \delta^{(k-1-m)}(M-m) \qquad \text{for } k \geqslant M+1. \tag{28}$$

In this way, finally, for $k = M + 1$, we find

$$\partial^{M+1} F_-(y,u)/\partial y^{M+1}|_{y=M} = (-1)^M B^M(u) \delta(0). \tag{29}$$

The above expression is not null since we have $\delta(x) = 0$ for $x \neq 0$, and $\delta(x) = \infty$ for $x = 0$ (Maisel 1971). Furthermore, for $k = M + p$ where $p = 1, 2, 3, \ldots$, we have

$$\partial^{M+p} F_-(y,u)/\partial y^{M+p}|_{y=M} = (-1)^M B^M(u) \delta^{(p-1)}(0) \qquad p = 1, 2, 3, \ldots.$$

Clearly, the function $F_-(y,u)$ is not polynomial for integer lethargy, because for ordinary polynomials of the type

$$P(y,u) = \sum_{n-0}^{M} A_n(u) y^n, \; \partial^{M+1} P(y,u)/\partial y^{M+1} = 0.$$

Consequently, for $y = M$

$$\partial^{M+1} P(y,u)/\partial y^{M+1}|_{y=M} \neq \partial^{M+1} F_-(y,u)/\partial y^{M+1}|_{y=M} \tag{30}$$

or, generally,

$$\partial^{M+p} P(y,u)/\partial y^{M+p}|_{y=M} \neq \partial^{M+p} F_-(y,u)/\partial y^{M+p}|_{y=M} \qquad \text{for } p = 1, 2, 3, \ldots.$$

Accordingly, the statement at the end of the comment is not correct, for any integer y.

Let us note that this analysis is irrelevant for the transcendental method theory, but it is interesting because the expression of the form

$$Y = \left\{ \frac{\sum_{m=0}^{M} (-B(u))^m (M-m)^m / m!}{\sum_{m=0}^{M+1} (-B(u))^m (M+1-m)^m / m!} \right\} \qquad \text{for } M \geqslant M_g \tag{31}$$

rests on the commentator's objection. Namely, this formula falsifies the commentator's claims even for finite $y = M$. In the above formula M_g is the value of the integer lethargy M when the error function defined as

$$G_-(M,u) = Y - B\exp(Y) \tag{32}$$

satisfies the inequality

$$P_-(M,u) = -\lg|G_-(M,u)| \geqslant P_m \qquad \text{for } M \geqslant M_g \tag{33}$$

where P_m is an arbitrary large positive number and, for large M, $1/P_m$ vanishes. Large M (large integer lethargy y) is defined in section 1, and Y is defined in equation (31).

Some calculations based on equations (31), (32) and (33) are shown in table 1 for various values of lethargy M, and applied to the parameter $B(u) = 0.17$, for instance.

Finally, having analysed all that is written in the comment, I must declare that the commentator's statements are neither significant nor relevant for the transcendental method theory and its application.

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